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APPLICATION OF ELASTIC-CURVE TRAVERSE METHOD TO MEMBERS HAVING STRAIGHT HAUNCHES

by

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INTRODUCTION

The method of analyzing certain types of indeterminate structures by traversing the elastic curves was first set forth in the Proceedings of the American Society of Civil Engineers of October 1934 by Mr. Ralph W. Stewart, Engineer of Structural Design for the City of Los Angeles, California. The original paper and the discussions which relate to it may be found in the Transactions of the A.S.C.E., Volume 101, 1936, pages 105 ff.

The application of the Elastic-Curve Traverse method to various types of structures was presented in a graduate thesis by Mr. Hjalmar Sundin at Rensselaer Polytechnic Institute in June of 1947. The authors are much endebted to Mr. Sundin for his work because of the clarification of the Traverse method it offered.

The guidance and suggestions of Dr. Joseph S, Kinney of the Civil Engineering Department, Rensselaer Polytechnic Institute, have been of assistance to the authors and were much appreciated.

OBJECT AND SCOPE

The general method of Elastic-Curve Traverses has been applied to co ntinuous beams, rigid frames, gabled frames, and other types of indeterminate structures composed of prismatic and non-prismatic members by Stewart, Drummond, Tsai, Floris, and others. Ho wever, the practical application of the method has been limited by the fact that no quick and accurate means of finding the areas of the M over EI diagrams and the location of the centers of gravity of the areas for non-prismatic member has been presented. This deficiency necessitated the finding of the areas and the centers of gravity by laborious integration. Stewart has stated that tables giving the M over EI areas for simple moments are unknown to him.

Mr. Walter Rupple has tables which have been used in applying the Elastic-Curve Taverse method, but it was felt that they could not be used in this paper for the following reasons:

1. The load coefficients are applicable to "fixed end moments," and the autho rs desired the M over EI area and the centers o f gravity of the areas for simply supported members.

2. The beam coefficients did not have the desired accuracy, nor were they set up in such a manner as to make their use quick or simple.

3. The method of computing the tables could not be determined, and it was assumed that approximate integration was used. The use of approximations deprives the method of the advantages of being a "classical" method.

The authors therefore felt that it would be valuable to devise a system by which the general Elastic-Curve Traverse method could be readily applied to the analysis of indeterminate structures made up of straight-haunched members subjected to concentrated loading and to end moment loading. The scope was limited to straight-haunched members because of the lack of available time.

The authors agree that the general Elastic-Curve method has the following advantages:

1. It gives a rough picture of what the deflected structure looks like(Austin H. Reeves, p. 116 Vol. 101, ASCE Trans.)

2. Does not require that formulas be memorized or copied, or complic ated sign conventions used(E.G. Paulet p. 116, Vol.101, ASCE Trans.).

3. Saves time in the construction of influence lines (Hjalmar Sundin, Master's Thesis, R.P.I., June 1947).

4. Simplifies and clarifies the distribution of sidesway moments(same as in 3.).

5. It is of great value in the analysis of multi-story and complex frames, because the method permits the writing of matrix equations in which the inverse, or reciprocal, of the a-matrix can be written.

The disadvantage of the method is that in problems involving non-prismatic members more time is consumed than when other methods are utilized. It is hoped that the system to be set forth by the authors in this paper will minimize if not eliminate this disadvantage.

The authors have derived formulas, calculated and arranged tables of values from the formulas for the area of the M over EI diagram and the location of the center of gravity of the area for straight-haunched members when subjected to concentrated loading and end-moment loading. The beam coefficient tables by Rupple were rearranged and recomputed using exact integratio n and four-place accuracy. These tables were coordinated into a simplified system which is illustrated in the example problems.

EXPLANATION OF THE ELASTIC-CURVE TRAVERSE METHOD

The Elastic-Curve Traverse method is based on the same assumptions as the other so-called classical methods:

1. The angular deflections are so small that the angles in radians, their sines, and their tangents may be considered equal without appreciable error.

2. The assumptions for the Theory of Continuity:

a. The neutral axes of all members at a joint meet in a point.

b. Distortions due to shear and direct stress are negligible.

c. The relative rotation of the two ends of a member, for short lengths of the axis, is proportional to the length over the moment of inertia.

d. Intensity and duration of loadings do not effect the elasticity of the members. Therefore, for short lengths of beam, the differential rotation of the two ends of the portion considered is directly proportionsl to the bending moment in the length considered.

3. Joints rotate, but do not have horizontal or vertical mo vement when sidesway is not considered.

The principles involved in the Elastic-Curve Traverse are:

1. The angles between the tangents at any two points on

the elastic curve of the flexed beam is equal to the area of the M over EI diagram between the two points considered.

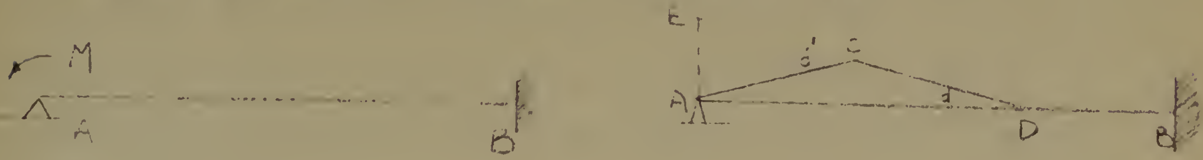
2. The curvature mentioned in 1. may be represented as an angle which in radians is numerically equal to the corresponding M over EI area. The angle is plotted opposite the center of gravity of the M over EI area.

3. For any unit of the M over EI diagram, a triangular traverse of the corresponding unit of the elastic curve can be constructed. This triangle is composed of the tangents and the chord o f the elastic curve. The angle between the tangents is defined in 2. Each angle between a tangent and the chord is directly proportional to the side opposite. This is because in beam flexure the angles are so small that they may be considered to be equal to their sines.

The first step in the application of the method is to determine the M over EI diagrams for the various loads. These diagrams are plotted on the tension side of the members. The angles which are numerically equal to the M over EI areas are placed at the center of gravity of the areas. When all of the angles have been placed, equations of continuity are used to obtain eqations which will relate the unknown quantities. The explanation of the method has purposely been kept to a bare minimum, because examples illustrate the method without the confusion of a wordy description. The following is a

simple illustration. More involved examples are included in the calculations. These samples also demonstrate the system the authors have devised to aid in analysing straight-haunched members.

The example to be used will be a propped cantilever, as shown in the sketch. A Prismatic beam is used.



This problem has been chosen, it gives a simple check on the method. It is known that the carryover factor for a prismatic beam is one-half. In other words the moment at the fixed end o f a propped cantilever will be one-half of the moment applied at the propped end. Thus, if the moment at the fixed end in the example turns out to be one-half of the applied moment, we will have a check on the method.

The solution is as follows:

1. The beam is prismatic, so the moment diagrams will be triangular with their centers of gravity at the third points.

2. Start the traverse at the fixed end, B. There is no slope or deflection at B, so the traverse follows the chord of the elastic curve to the center of gravity of the M over EI diagram due to the moment at B. At this point, D, put an angle equal to M_B L over 2 EI. Let this angle be called d.

If this angle were applied alone it would cause a deflection equal to 2/3Ld at A.

3. Continue the traverse along the line DE to the center o f gravity of the are due to the moment at A. At this point, C, plot an angle eoual to $M_A L$ over 2EI. Because point A can not move, the traverse must go through A. The angle at C must therefore cause a deflection of the traverse equal to AE. But the deflec tion due to d', the angle at C, must also be equal to 1/3Ld'.

4. There is one unknown, M_B, and this may be found because by the conditions of the problem there is no deflection or movement of the points A and B. This is the same as saying that the traverse offsets due to d and d' must be equal.

$$2/3Ld = 1/3Ld' \quad \text{There fore} \quad 2d = d'$$

$$\text{and } 2M_B = M_A$$

5. The results check with the predictions. This gives a check on the method.

This brief explanation and example of the Elastic-Curve Traverse meth od can be supplemented by refering to Sundin or Stewart(see Bibliography), who have investigated the method's application to various types of structures.

METHOD OF CALCULATING AREA AND CENTER OF GRAVITY TABLES

It was necessary to have tables which gave the M over EI area and the location of the center of gravity of the area for simply supported straight-haunched non-prismatic members in order that the system devised by the authors could be used. This type of beam is used frequently in concrete and steel work.

The modulus of elasticity has been omitted from the calculations, because the material from which the member is made is assumed to be homogeneous and would thus have a constant E throughout.

Known simple moments and moment diagrams were placed on the members. The ordinates of these diagrams were divided by the moment of inertia, I, at particular sections, and then the M over EI were summed up over the entire length of the member by integration. Hereafter, the area will be called the M over I area, because the E has been left out of the calculations. Exact integration was used to keep the method free from approximations and to give the desired degree of accuracy. The summation of the M over I values is the area desired.

Next, the M over I values at particular sections were multiplied by their distances from one end, and the products were integrated to give the total moment of the M over I diagram about the end. This summation was divided by the previously calculated area to give the distance of the center of gravity of the area from the end of the member.

The calculations were made for a unit length of member. This was possible because the length appears in all of the terms of the area and the center of gravity expressions- to the sec ond power in the area and to the third power in the center of gravity.

The maximum ordinate of moment diagram also appears in all of the terms, so it was factored out. This is the same as saying that the tables were made up for a unit maximum mo ment.

It was found to be more convenient to make up the tables using the ratio of the maximum to the minimum depth of member, rather than the ratio of the moments of inertia as Rupple do es.

It was felt that the most important variables were the length o f haunch, the load point, and the ratio of the maximum to minimum depth, so these were made the values with which the tables are entered.

A more complete range o f values, possibly enough to make up a set of curves, would be very valuable, but the authors were limited as to the time available and could no t carry out such an extensive program.

METHOD OF USING AREA AND CENTER OF GRAVITY TABLES

For Concentrated loading.

To find the area of the M over I diagram, the proper

table is entered with the ratio of the maximum to minimum depth, the ratio of the d istance to the load to the length of the member, and the ratio of the length of haunch to the length o f the member. Using these three values, a coefficient is picked from the table. This is called the load coefficient, k. The area of the M over I diagram is equal to kLM over the minimum depth cubed where M is the maximum simple moment caused by the concentrated loading, L is the actual length of the member, and k is the load coefficient.

To find the center of gravity of the M over I diagram, enter the table with the same values as for the area. A factor k' is pick ed from the table and multiplied by the actual length of the member. This gives the distance of the center of gravity from the left end of the member.

For a M oment Applied at the End of the Member.

To find the area, enter the table with the ratio of the maximum to minimum depth and the ratio of the length of haunch to the to tal length of member and pick out a coefficient k''. The area of the M over I diagram will then be k''LM ov er the minimum depth cubed. The symbols have the same meaning as in the Concentrated Load tables.

To find the center of gravity, the coefficient is picked using the same values as for the area. This coefficient,k''', is multiplied by the actual length of the member to give the distance to the center of gravity from the left end of the member.

Each table is accompanied by a sketch. It is important that the user examine the tables and the sketches thoroughly to make certain that he has taken the entering values correctly. Particular care must be taken to make sure that the distance to the center of gravity is from the proper end.

Interpolation may be used with the authors' tables, but to be accurate, values should be substituted into the original equations which are included in this paper. Interpolation is straight line, whereas the values in the table follow a curve. The variation is particularly large at the extremities o f the tables, making interpolation in these portions inadvisable.

SUMMARY AND RECOMMENDATIONS CONCERNING FUTURE WORK

It is desired to present here a summary of the work d one by the authors and to state the possible extension of the application of the Traverse method and to restate the advantages of the method.

The authors have filled the gap which existed between the theoretical and the practical consideration of straight-haunched members. This was done by devising a system by which the Traverse method could be easily applied to non-prismatic members with accuracy and by calculating tables which were necessary to make the system work. These tables include: load coefficients for finding the M over I areas for concentrated loads and for end moments for members with haunches at one or both ends; and distance factors for finding the distance from one end to the center of gravity

for the same loadings and types of members as for the area.

The advantages of the Elastic-Curve Traverse method can be summarized as follows:

1. An exact solution is obtained, the accuracy of which is dependent only upon the accuracy of the mathmatical substitutions in the formulas presented by the authors. This statement applies only to the types of members considered in this paper, but when an extension of the work is done, it will be true of all types o f members.

2. A rough picture of the deflected structure results when the traverse is completed.

3. Successful utilization of the method does not depend on the memorization of formulas or complicated sign conventions.

4. Simplification and clarification of sidesway, influence lines, and end fixity are outstanding advantages of the Traverse method. Sundin has considered these matters in detail, so they were not reconsidered in this paper. It is mentioned here to show that the method has advantages other than those which are apparent in this thesis.

5. The ease o f solution of complicated structures when using the tables, particularly with symetrical loading and symetrical members, is advantageous. It is thought that the Traverse method is better than Moment-Distribution for ordinary structures with symetrical loading. This is the authors' opinio n formulated while working sample problems and checking them by the Moment-Distribution method.

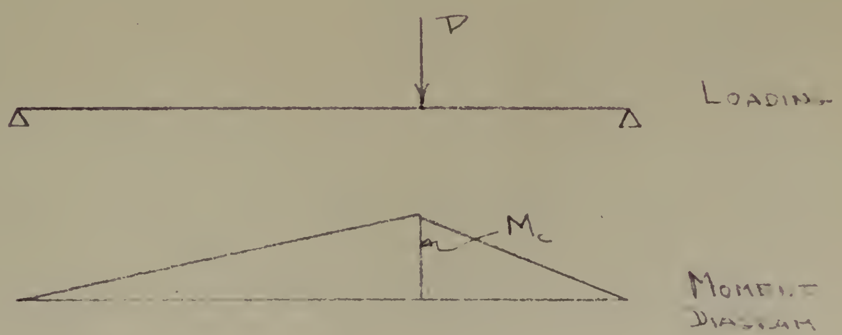
Investigation into the applicability of the Traverse

method to the analysis of arches might prove valuable.

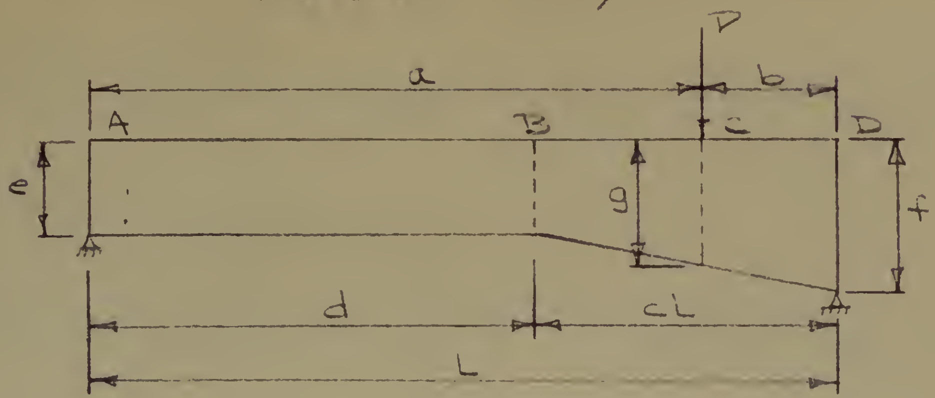
It is thought that future work on the Elastic-Curve traverse method should include the calculation of tables giving the areas of the M over EI diagrams for straight and parabolic haunched members subjected to uniform loads when simply supported and for parabolic haunched members with concentrated loads and for the distances to the centers of gravity of the above areas. It is felt that particular emphasis should be placed on the calculation of the above mentioned tables, because therein lies the real practical usefulness of the Elastic-Curve Traverse method.

Derivation of Formulae for Determination of Area and Moment of Area of M/I - Diagrams

In all determinations the beam will be considered to be simple supported at both ends. The value "M_c" will be considered as the maximum value of the simple moment diagram with concentrated loading applied and the value of "M_M" will be the maximum value of the simple moment with end moment applied at one end or the other of the beam. These values are indicated in the sketches below.



CASE I-A: MEMBER WITH STRAIGHT HAUNCH AT ONE END. $cL > b$



FOR DETERMINING THE AREA: $A = \int_A^B \frac{M}{I}\,dx + \int_B^C \frac{M}{I}\,dx + \int_C^D \frac{M}{I}\,dx$

SECTION	X=0 AT	X→	M	M/I	LIMITS
AB	A	A-B	$\frac{M_c x}{a}$	$\frac{12 M_c x}{a e^3}$	0-d
BC	B	B-C	$\frac{M_c(d+x)}{a}$	$\frac{12 M_c (d+x)}{a\left[e+\left(\frac{g-e}{a-d}\right)x\right]^3}$	0-(a-d)
CD	D	D-C	$\frac{M_c x}{b}$	$\frac{12 M_c x}{b\left[f-\left(\frac{f-g}{b}\right)x\right]^3}$	0-b

INTEGRATING EACH SECTION SEPERATELY:

SECTION AB $\quad \int_A^B \frac{M}{I}\,dx = \int_0^d \frac{12 M_c x\,dx}{a e^3} = \frac{6 M_c d^2}{a e^3}$

SECTION BC $\quad \int_B^C \frac{M}{I}\,dx = \int_0^{a-d} \frac{12 M_c (d+x)\,dx}{a\left[e+\left(\frac{g-e}{a-d}\right)x\right]^3}$

$$= \frac{12 M_c d}{a} \int_0^{a-d} \frac{dx}{\left[e+\left(\frac{g-e}{a-d}\right)x\right]^3} + \frac{12 M_c}{a} \int_0^{a-d} \frac{x\,dx}{\left[e+\left(\frac{g-e}{a-d}\right)x\right]^3}$$

$$= \frac{12M_c d}{a}\left[-\frac{1}{2\left(\frac{g-e}{a-d}\right)g^2} + \frac{1}{2\left(\frac{g-e}{a-d}\right)e^2}\right]$$

$$+ \frac{12M_c}{a}\left(\frac{a-d}{g-e}\right)^2\left[-\frac{1}{g} + \frac{e}{2g^2} + \frac{1}{e} - \frac{1}{2e}\right] = \frac{6M_c d}{a}\left[\frac{(a-d)(g+e)}{e^2 g^2}\right]$$

$$+ \frac{6M_c}{a}\left[\frac{(a-d)^2}{e g^2}\right]$$

SECTION CD

$$\int_C^D \frac{M}{I}\,dx = \int_0^b \frac{12M_c x\,dx}{b\left[f-\left(\frac{f-g}{b}\right)x\right]^3}$$

$$= \frac{12M_c}{b}\left[\frac{b^2}{(f-g)^2}\right]\left[-\frac{1}{g} + \frac{f}{2g^2} + \frac{1}{f} - \frac{1}{2f}\right] = \frac{6M_c b}{f g^2}$$

SUMMING UP THE SECTIONS:

$$\text{TOTAL } A = \frac{6M_c d^2}{ae^3} + \frac{6M_c d}{a}\left[\frac{(a-d)(g+e)}{e^2 g^2}\right]$$

$$+ \frac{6M_c}{a}\left[\frac{(a-d)^2}{e g^2}\right] + \frac{6M_c b}{f g^2}$$

FOR DETERMINATION OF THE CENTER OF GRAVITY OF THE AREA OF THE M/I - DIAGRAM, THE TOTAL MOMENT OF THE AREA ABOUT THE LEFT END OF THE BEAM IS FOUND.

$$\text{TOTAL MOMENT } M = \int_A^B \frac{Mx\,dx}{I} + \int_B^C \frac{Mx\,dx}{I} + \int_C^D \frac{Mx\,dx}{I}$$

Section	X=0 at	X→	$\frac{M}{I}$	$\frac{Mx}{I}$	Limits
AB	A	A-B	$\frac{12M_c x}{a e^3}$	$\frac{12M_c x^2}{a e^3}$	$0-d$
BC	B	B-C	$\frac{12M_c(d+x)}{a\left[e+\left(\frac{g-e}{a-d}\right)x\right]^3}$	$\frac{12M_c(d+x)^2}{a\left[e+\left(\frac{g-e}{a-d}\right)x\right]^3}$	$0-(a-d)$
DC	D	D-C	$\frac{12M_c x}{b\left[f-\left(\frac{f-g}{b}\right)x\right]^3}$	$\frac{12M_c x(L-x)}{b\left[f-\left(\frac{f-g}{b}\right)x\right]^3}$	$0-b$

AGAIN INTEGRATING EACH SECTION SEPARATELY:

SECTION AB
$$\int_A^B \frac{Mx\,dx}{I} = \frac{12M_c}{ae^3}\int_0^d x^2\,dx = \frac{4M_c d^3}{ae^3}$$

SECTION BC
$$\int_B^C \frac{Mx\,dx}{I} = \frac{12M_c}{a}\int_0^{a-d} \frac{(d^2+2dx+x^2)\,dx}{\left[e+\left(\frac{g-e}{a-d}\right)x\right]^3}$$

$$= \frac{6M_c d^2}{a}\left[\frac{(a-d)(g+e)}{e^2g^2}\right] + \frac{12M_c d}{a}\left[\frac{(a-d)^2}{eg^2}\right]$$

$$+ \frac{12M_c}{a}\left(\frac{a-d}{g-e}\right)^3\left[\ln g + \frac{2e}{g} - \frac{e^2}{2g^2} - \ln e - 2 + \frac{1}{2}\right]$$

$$= \frac{6M_c d^2}{a}\left[\frac{(a-d)(g+e)}{e^2g^2}\right] + \frac{12M_c d}{a}\left[\frac{(a-d)^2}{eg^2}\right]$$

$$+ \frac{12M_c}{a}\left(\frac{a-d}{g-e}\right)^3\left[\ln\frac{g}{e} + \frac{2e}{g} - \frac{e^2}{2g^2} - \frac{3}{2}\right]$$

SECTION CD
$$\int_C^D \frac{Mx\,dx}{I} = \int_0^b \frac{12M_c L x\,dx}{b\left[f-\left(\frac{f-g}{b}\right)x\right]^3}$$

$$- \int_0^b \frac{12M_c x^2\,dx}{b\left[f-\left(\frac{f-g}{b}\right)x\right]^3} = \frac{12M_c L}{b}\left[\frac{b^2}{(f-g)^2}\right]\left[-\frac{1}{g} + \frac{f}{2g^2} + \frac{1}{f} - \frac{1}{2f}\right]$$

$$+ \frac{12 M_c}{b}\left[\frac{b^3}{(f-g)^3}\right]\left[\ln g + \frac{2f}{g} - \frac{f^2}{2g^2} - \ln f - 2 + \frac{1}{2}\right]$$

$$= \frac{6 M_c b L}{f g^2} + \frac{12 M_c b^2}{(f-g)^3}\left[\ln \frac{g}{f} + \frac{2f}{g} - \frac{f^2}{2g^2} - \frac{3}{2}\right]$$

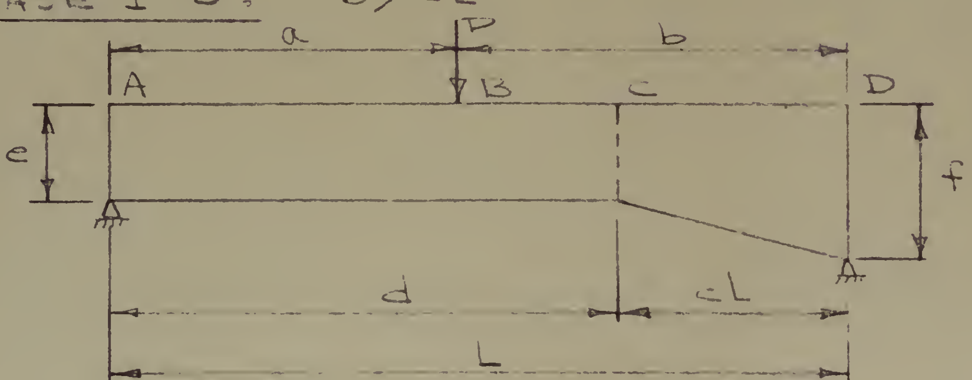
SUMMING UP THE SECTIONS:

$$\text{TOTAL MOMENT } M = \frac{4 M_c d^3}{a e^3} + \frac{6 M_c d^2}{a}\left[\frac{(a-d)(g+e)}{e^2 g^2}\right]$$

$$+ \frac{12 M_c d}{a}\left[\frac{(a-d)^2}{e g^2}\right] + \frac{12 M_c}{a}\left(\frac{a-d}{g-e}\right)^3\left[\ln \frac{g}{e} + \frac{2e}{g} - \frac{e^2}{2g^2} - \frac{3}{2}\right]$$

$$+ \frac{6 M_c L b}{f g^2} + \frac{12 M_c b^2}{(f-g)^3}\left[\ln \frac{g}{f} + \frac{2f}{g} - \frac{f^2}{2g^2} - \frac{3}{2}\right]$$

CASE I-B: $b > cL$



SECTION	X=0	X →	M	M/I	LIMITS
AB	A	A-B	$\frac{M_c x}{a}$	$\frac{12 M_c x}{a e^3}$	$0 - a$
BC	B	B-C	$\frac{M_c(b-x)}{b}$	$\frac{12 M_c (b-x)}{b e^3}$	$0-(b-cL)$
CD	D	D-C	$\frac{M_c x}{b}$	$\frac{12 M_c x}{b\left[f-\left(\frac{f-e}{cL}\right)x\right]^3}$	$0 - cL$

INTEGRATING EACH SECTION SEPERATELY:

SECTION AB $\displaystyle\int_0^a \frac{12 M_c x\, dx}{a e^3} = \frac{6 M_c a}{e^3}$

SECTION BC $\displaystyle\int_0^{b-cL} \frac{12 M_c (b-x)\, dx}{b e^3}$

$$= \frac{12 M_c}{b e^3}\left[b^2 - bcL - \frac{b^2 + 2bcL - c^2L^2}{2} \right]$$

$$= \frac{12 M_c}{b e^3}\left[\frac{2b^2 - 2bcL - b^2 + 2bcL - c^2L^2}{2} \right] = \frac{6 M_c}{b e^3}\left[b^2 - c^2L^2 \right]$$

SECTION CD $\displaystyle\int_0^{cL} \frac{12 M_c x\, dx}{b\left[f - \left(\frac{f-e}{cL}\right)x\right]^3}$

$$= \frac{12 M_c}{b}\left[\frac{c^2L^2}{(f-e)^2}\right]\left[-\frac{1}{e} - \frac{f}{2e^2} + \frac{1}{f} - \frac{f}{2f^2}\right] = \frac{6 M_c}{b}\left[\frac{c^2L^2}{fe^2}\right]$$

SUMMING UP THE SECTIONS:

$$\text{TOTAL } A = 6 M_c \left[\frac{a}{e^3} + \frac{1}{b e^3}\left(b^2 - c^2L^2\right) + \frac{c^2L^2}{b f e^2} \right]$$

FOR DETERMINING TOTAL MOMENT M:

SECTION	X=0	X →	M/I	Mx/I	LIMITS
AB	A	A-B	$\frac{12 M_c x}{a e^3}$	$\frac{12 M_c x^2}{a e^3}$	$0 - a$
BC	B	B-C	$\frac{12 M_c (b-x)}{b e^3}$	$\frac{12 M_c (b-x)(a+x)}{b e^3}$	$0 - (b - cL)$
CD	D	D-C	$\frac{12 M_c x}{b\left[f - \left(\frac{f-e}{cL}\right)x\right]^3}$	$\frac{12 M_c x (L-x)}{b\left[f - \left(\frac{f-e}{cL}\right)x\right]^3}$	$0 - cL$

INTEGRATING BY SECTIONS:

SECTION AB
$$\int_0^a \frac{12 M_c x^2 dx}{a e^3} = \frac{4 M_c a^2}{e^3}$$

SECTION BC
$$\int_0^{b-cL} \frac{12 M_c \left[ab + (b-a)x - x^2\right] dx}{b e^3}$$
$$= \frac{12 M_c}{b e^3}\left[ab(b-cL) + \frac{(b-a)(b-cL)^2}{2} - \frac{(b-cL)^3}{3}\right]$$

SECTION CD
$$\int_0^{cL} \frac{12 M_c (xL - x)\, dx}{b\left[f - \left(\frac{f-e}{cL}\right)x\right]^3}$$

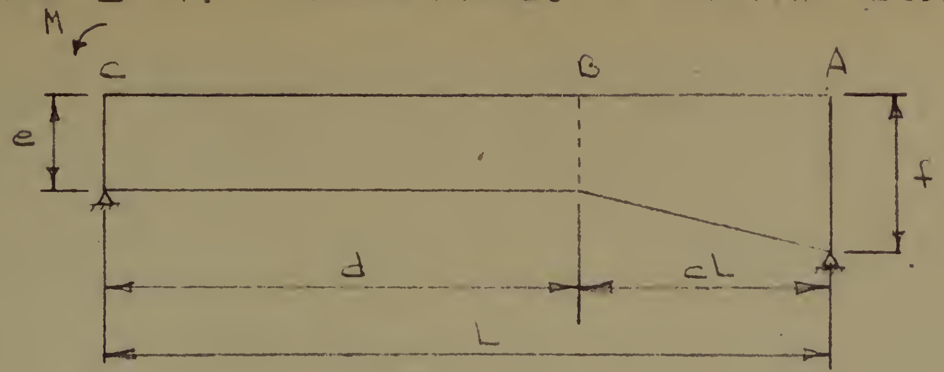
$$= \frac{6 M_c}{b}\left[\frac{c^2 L^3}{f e^2}\right] + \frac{12 M_c}{b}\left[\frac{c^3 L^3}{(f-e)^3}\right]\left[\ln\frac{e}{f} + \frac{2f}{e} - \frac{f^2}{2e^2} - \frac{3}{2}\right]$$

SUMMING UP THE SECTIONS:

$$\text{TOTAL MOMENT } M = \frac{4 M_c a^2}{e^3}$$
$$+ \frac{12 M_c}{b e^3}\left[ab(b-cL) + \frac{(b-a)(b-cL)^2}{2} - \frac{(b-cL)^3}{3}\right]$$
$$+ \frac{6 M_c c^2 L^3}{b f e^2} + \frac{12 M_c}{b}\left[\frac{c^3 L^3}{(f-e)^3}\right]\left[\ln\frac{e}{f} + \frac{2f}{e} - \frac{f^2}{2e^2} - \frac{3}{2}\right]$$

NEXT THE CASES FOR WHICH SIMPLE MOMENTS ARE APPLIED TO EITHER END OF BEAMS WITH A STRAIGHT HAUNCH AT ONE END WILL BE CONSIDERED. AGAIN THE BEAMS ARE SIMPLY SUPPORTED.

CASE II-A: MOMENT APPLIED AT STRAIGHT END



FOR DETERMINING AREA:

SECTION	X=0	X→	M	M/I	LIMITS
AB	A	A-B	$\frac{M_M x}{L}$	$\frac{12 M_M x}{L\left[f-\left(\frac{f-e}{cL}\right)x\right]^3}$	0-cL
BC	B	B-C	$\frac{M_M(cL+x)}{L}$	$\frac{12 M_M (cL+x)}{Le^3}$	0-(L-cL)

INTEGRATING EACH SECTION SEPERATELY:

SECTION AB $\displaystyle\int_0^{cL} \frac{12 M_M x\,dx}{L\left[f-\left(\frac{f-e}{cL}\right)x\right]^3} = \frac{12 M_M}{L}\int_0^{cL}\frac{x\,dx}{\left[f-\left(\frac{f-e}{cL}\right)x\right]^3}$

$$= \frac{12 M_M}{L}\left[\frac{c^2L^2}{(f-e)^2}\right]\left[-\frac{1}{e}+\frac{f}{2e}+\frac{1}{f}-\frac{1}{2f}\right] = \frac{6 M_M c^2 L}{fe^2}$$

SECTION BC $\displaystyle\int_0^{L-cL}\frac{12 M_M (cL+x)\,dx}{Le^3}$

$$= \frac{12 M_M cL(L-cL)}{Le^3} + \frac{6 M_M (L-cL)^2}{Le^3}$$

$$= \frac{6 M_M}{Le^3}\left[2cL^2-2c^2L^2+L^2-2cL^2+c^2L^2\right] = \frac{6 M_M L(1-c^2)}{e^3}$$

SUMMING UP THE SECTIONS:

$$\text{TOTAL AREA } A = \frac{6M_mL}{e^2}\left[\frac{c^2}{f} + \frac{(1-c^2)}{e}\right]$$

FOR DETERMINING THE MOMENT OF THE AREA, THIS TIME ABOUT THE RIGHT END FOR EASE OF INTEGRATION:

SECTION	X=O	X	M/I	Mx/I	LIMITS
AB	A	A-B	$\dfrac{12M_mx}{L\left[f-\left(\frac{f-e}{cL}\right)x\right]^3}$	$\dfrac{12M_mx^2}{L\left[f-\left(\frac{f-e}{cL}\right)x\right]^3}$	$0-cL$
BC	B	B-C	$\dfrac{12M_m(cL+x)}{Le^3}$	$\dfrac{12M_m(cL+x)^2}{Le^3}$	$0-(L-cL)$

AGAIN INTEGRATING BY SECTIONS:

SECTION AB $\displaystyle\int_0^{cL}\frac{12M_mx^2\,dx}{L\left[f-\left(\frac{f-e}{cL}\right)x\right]^3}$

$$= \frac{12M_m}{L}\left[-\frac{c^3L^3}{(f-e)^3}\right]\left[\ln e + \frac{2f}{e} - \frac{f^2}{2e^2} - \ln f - 2 + \frac{1}{2}\right]$$

$$= -\frac{12M_m c^3L^2}{(f-e)^3}\left[\ln\frac{e}{f} + \frac{2f}{e} - \frac{f^2}{2e^2} - \frac{3}{2}\right]$$

SECTION BC $\displaystyle\int_0^{L-cL}\frac{12M_m(c^2L^2+2cL+x^2)\,dx}{Le^3}$

$$= \frac{12M_m}{Le^3}\left[c^2L^2(L-cL) + cL(L-cL)^2 + \frac{(L-cL)^3}{3}\right]$$

$$= \frac{12M_mL^2}{e^3}(1-c)\left(\frac{3c^2+3c-3c^2+1-2c+c^2}{3}\right)$$

$$= \frac{4M_mL^2}{e^3}(1-c)(1+c+c^2)$$

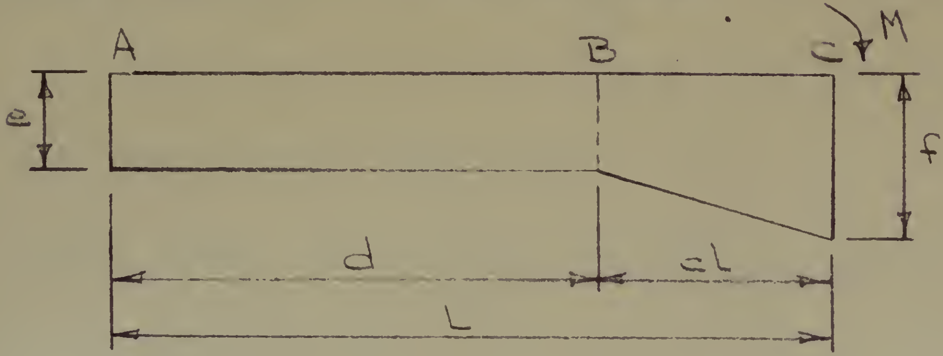
SUMMING UP THE SECTIONS:

TOTAL MOMENT ABOUT THE RIGHT END OF THE BEAM

$$M = \frac{4M_M L^2}{e^3}(1-c)(1+c+c^2)$$

$$- \frac{12 M_M c^3 L^2}{(f-e)^3}\left[\ln\frac{e}{f} + \frac{2f}{e} - \frac{f^2}{2e^2} - \frac{3}{2}\right]$$

CASE II-B: MOMENT APPLIED AT HAUNCHED END



FOR DETERMINING THE AREA:

SECTION	X=0	X→	M	$\frac{M}{I}$	LIMITS
AB	A	A-B	$\frac{M_M x}{L}$	$\frac{12 M_M x}{Le^3}$	0-d
BC	B	B-C	$\frac{M_M(d+x)}{L}$	$\frac{12 M_M(d+x)}{L\left[e+\left(\frac{f-e}{L-d}\right)x\right]^3}$	0-(L-d)

INTEGRATING BY SECTIONS:

SECTION AB $$\int_0^d \frac{12 M_M x\,dx}{Le^3} = \frac{6 M_M d^2}{Le^3}$$

SECTION BC $\displaystyle\int_0^{L-d} \frac{12M_M(d+x)\,dx}{L\left[e+\left(\frac{f-e}{L-d}\right)x\right]^3}$

$$= \frac{12M_M d}{L}\left[-\frac{1}{2\frac{(f-e)}{(L-d)}f^2} + \frac{1}{2\frac{(f-e)}{(L-d)}e^2}\right]$$

$$+ \frac{12M_M}{L}\left(\frac{L-d}{f-e}\right)^2\left[-\frac{1}{f} + \frac{e}{2f^2} + \frac{1}{e} - \frac{1}{2e}\right]$$

$$= \frac{6M_M d}{L}\left[\frac{(L-d)(f+e)}{f^2e^2}\right] + \frac{6M_M(L-d)^2}{Lf^2e}$$

SUMMING UP THE SECTIONS:

$$\text{TOTAL AREA } A = \frac{6M_M d^2}{Le^3} + \frac{6M_M d\,(L-d)(f+e)}{L\,f^2e^2} + \frac{6M_M(L-d)^2}{Lf^2e}$$

FOR DETERMINING THE MOMENT OF THE AREA, AGAIN ABOUT THE LEFT END OF THE BEAM:

SECTION	X=0	x →	M/I	Mx/I	LIMITS
AB	A	A-B	$\frac{12M_M x}{Le^3}$	$\frac{12M_M x^2}{Le^3}$	0 - d
BC	B	B-C	$\frac{12M_M(d+x)}{L\left[e+\left(\frac{f-e}{L-d}\right)x\right]^3}$	$\frac{12M_M(d+x)^2}{L\left[e+\left(\frac{f-e}{L-d}\right)x\right]^3}$	0-(L-d)

INTEGRATING BY SECTIONS:

SECTION AB $\displaystyle\int_0^{d} \frac{12M_M x^2\,dx}{Le^3} = \frac{4M_M d^3}{Le^3}$

SECTION BC $$\int_0^{L-d} \frac{12 M_M (d+x)^2\, dx}{L\left[e+\left(\frac{f-e}{L-d}\right)x\right]^3}$$

$$= \frac{12 M_M d^2}{L}\int_0^{L-d} \frac{dx}{\left[e+\left(\frac{f-e}{L-d}\right)x\right]^3} + \frac{24 M_M d}{L}\int_0^{L-d} \frac{x\, dx}{\left[e+\left(\frac{f-e}{L-d}\right)x\right]^3}$$

$$+ \frac{12 M_M}{L}\int_0^{L-d} \frac{x^2\, dx}{\left[e+\left(\frac{f-e}{L-d}\right)x\right]^3} = \frac{6 M_M d^2 (L-d)(f+e)}{L f^2 e^2}$$

$$+ \frac{12 M_M d (L-d)^2}{L f^2 e} + \frac{12 M_M}{L}\left(\frac{L-d}{f-e}\right)^3\left[\ln\frac{f}{e} + \frac{2e}{f} - \frac{e^2}{2f^2} - \frac{3}{2}\right]$$

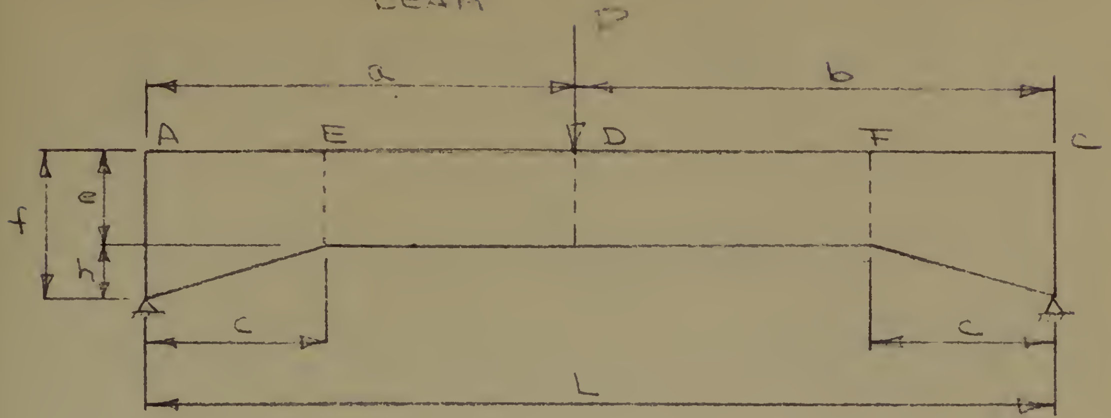
SUMMING UP THE SECTIONS:

TOTAL MOMENT OF THE AREA ABOUT THE

LEFT END $$M = \frac{4 M_M d^3}{L e^3} + \frac{6 M_M d^2 (L-d)(f+e)}{L f^2 e^2}$$
$$+ \frac{12 M_M d (L-d)^2}{L f^2 e} + \frac{12 M_M}{L}\left(\frac{L-d}{f-e}\right)^3\left[\ln\frac{f}{e} + \frac{2e}{f} - \frac{e^2}{2f^2} - \frac{3}{2}\right]$$

THIS CONCLUDES THE DETERMINATION OF FORMULAS FOR FINDING THE AREA AND THE MOMENT OF THE AREA OF THE M/I - DIAGRAMS FOR BEAMS WITH A HAUNCH AT ONE END ONLY. NEXT, BEAMS WITH SYMMETRICAL HAUNCHES AT EITHER END WILL BE CONSIDERED.

Case III - A   Load on Prismatic Section of Beam



For simplification let $j = c(h+e) = cf$
and $g = L - c$

Then for determining the area:

Section	$x=0$	$x \rightarrow$	M	$\frac{M}{I}$	Limits
AE	A	A-E	$\frac{M_c x}{a}$	$\frac{12 M_c c^3 x}{a(j-hx)^3}$	$0-c$
ED	E	E-D	$\frac{M_c(c+x)}{a}$	$\frac{12 M_c(c+x)}{ae^3}$	$0-(a-c)$
DF	D	D-F	$\frac{M_c(b-x)}{b}$	$\frac{12 M_c(b-x)}{be^3}$	$0-(b-c)$
FC	F	F-C	$\frac{M_c(c-x)}{b}$	$\frac{12 M_c c^3(c-x)}{b(ec+hx)^3}$	$0-c$

Integrating by sections:

Section AE
$$\int_0^c \frac{12 M_c c^3 x\,dx}{a(j-hx)^3} = \frac{12 M_c c^3}{ah^2}\left[\frac{2hc-j}{2(j-hc)^2}+\frac{1}{2j}\right]$$

$$= \frac{6 M_c c^2}{afe^2}$$

SECTION ED $\displaystyle\int_0^{a-c} \frac{12M_c(c+x)\,dx}{ae^3}$

$$= \frac{12M_c}{ae^3}\left[c(a-c)+\frac{(a-c)^2}{2}\right] = \frac{6M_c}{ae^3}\left[a^2-c^2\right]$$

SECTION DF $\displaystyle\int_0^{b-c} \frac{12M_c(b-x)\,dx}{be^3}$

$$= \frac{12M_c}{be^3}\left[b(b-c)-\frac{(b-c)^2}{2}\right] = \frac{6M_c(b^2-c^2)}{be^3}$$

SECTION FC $\displaystyle\int_0^{c} \frac{12M_c c^3(c-x)\,dx}{b(ec+hx)^3}$

$$= \frac{12M_c c^3}{b}\left[\frac{ec+hc}{2h^2(ec+hc)^2}+\frac{hc-ec}{2h^2(ec)^2}\right] = \frac{6M_c c^2}{bfe^2}$$

SUMMING UP THE SECTIONS:

TOTAL AREA $\displaystyle A = \frac{6M_c c^2}{afe^2} + \frac{6M_c(a^2-c^2)}{ae^3}$

$$+\frac{6M_c(b^2-c^2)}{be^3} + \frac{6M_c c^2}{bfe^2}$$

FOR DETERMINING THE MOMENT OF THE AREA ABOUT THE LEFT END OF THE BEAM, THE FOLLOWING TABLE WILL BE SET UP.

SECTION	X=0	X →	M/I	Mx/I	LIMITS
AE	A	A-E	$\frac{12M_c c^3 x}{a(j-hx)^3}$	$\frac{12M_c c^3 x^2}{a(j-hx)^3}$	$0-c$
ED	E	E-D	$\frac{12M_c(c+x)}{ae^3}$	$\frac{12M_c(c+x)^2}{ae^3}$	$0-(a-c)$
DF	D	D-F	$\frac{12M_c(b-x)}{be^3}$	$\frac{12M_c(b-x)(a+x)}{be^3}$	$0-(b-c)$
FC	F	F-C	$\frac{12M_c c^3(c-x)}{b(ec+hx)^3}$	$\frac{12M_c c^3(c-x)(L-c+x)}{b(ec+hx)^3}$	$0-c$

INTEGRATING BY SECTIONS:

SECTION AE $\displaystyle\int_0^c \frac{12M_c c^3 x^2\,dx}{a(j-hx)^3}$

$$= \frac{12M_c c^3}{ah^3}\left[\ln\left(\frac{j}{j-hc}\right) - \frac{(2j^2-4jhc)}{2(j-hc)^2} + \frac{3}{2}\right]$$

$$= \frac{12M_c c^3}{ah^3}\left[\ln\frac{j}{ec} - \frac{(4e-f)f}{2e^2} + \frac{3}{2}\right]$$

SECTION ED $\displaystyle\int_0^{a-c} \frac{12M_c(c+x)^2\,dx}{ae^3}$

$$= \frac{12M_c}{ae^3}\left[\frac{(c+a-c)^3-c^3}{3}\right] = \frac{4M_c}{ae^3}\left[a^3-c^3\right]$$

SECTION DF $\displaystyle\int_0^{b-c} \frac{12M_c(b-x)(a+x)\,dx}{be^3}$

$$= \frac{12M_c}{be^3}(b-c)\left[ab + \frac{(b-a)(b-c)}{2} - \frac{(b-c)^2}{3}\right]$$

$$= \frac{2M_c}{be^3}(b-c)\left[3ab+b^2+bc+3ac-2c^2\right]$$

SECTION FC $\displaystyle\int_0^c \frac{12M_c c^3(c-x)(g+x)\,dx}{b(ec+hx)^3}$

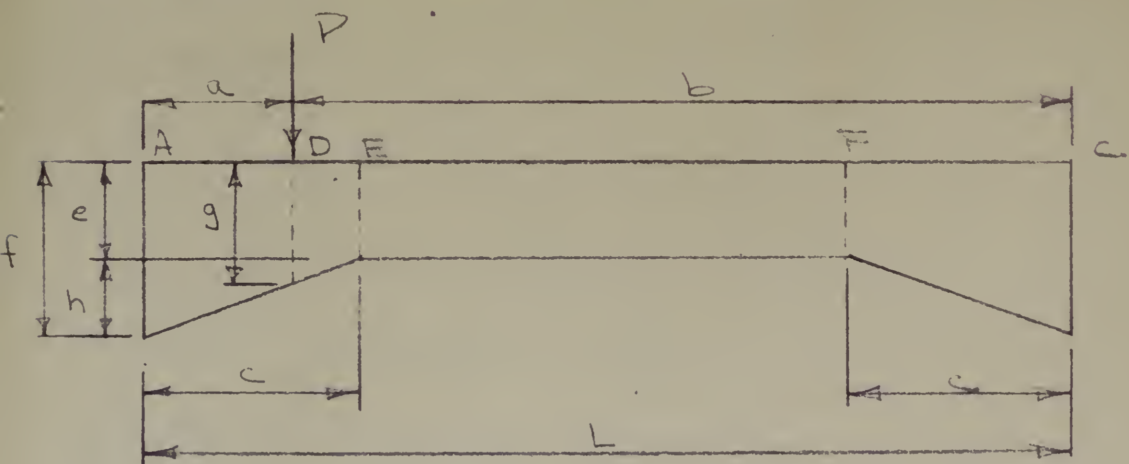
$$= \frac{12 M_c c^3}{b}\left[\frac{-cg}{2h(ec+ch)^2} + \frac{cg}{2e^2c^2h} + \frac{c-g}{h^2}\left(-\frac{2hc+ec}{2(ec+hc)^2} + \frac{ec}{2e^2c^2}\right) - \frac{1}{h^3}\left(\ln fc + \frac{3e^2c^2+4ec^2h}{2(ec+hc)^2} - \ln ec - \frac{3e^2c^2}{2e^2c^2}\right)\right]$$

$$= \frac{12 M_c c^3}{b}\left[\frac{g}{2c}\left(\frac{f+e}{e^2f^2}\right) + \frac{c-g}{2cf^2e} + \frac{1}{h^3}\left(\ln\frac{e}{f} - \frac{4ef-e^2}{2f^2} + \frac{3}{2}\right)\right]$$

SUMMING UP THE SECTIONS, THE TOTAL MOMENT ABOUT THE LEFT END OF THE BEAM IS:

$$M = \frac{12 M_c c^3}{ah^3}\left[\ln\frac{j}{ec} - \frac{(4e-f)f}{2e^2} + \frac{3}{2}\right] + \frac{4M_c}{ae^3}\left[a^3-c^3\right] + \frac{12 M_c}{be^3}(b-c)\left[3ab+b^2+bc+3ac-2c^2\right] + \frac{12 M_c c^3}{b}\left[\frac{g}{2c}\left(\frac{f+e}{e^2f^2}\right) + \frac{c-g}{2cf^2e} + \frac{1}{h^3}\left(\ln\frac{e}{f} - \frac{4ef-e^2}{2f^2} + \frac{3}{2}\right)\right]$$

CASE III-B LOAD ON HAUNCHED SECTION



FOR SIMPLIFICATION LET: $d = c - a$

$j = hd + ec$

$g = L - c$

FOR DETERMINING THE AREA:

SECTION	X=0	X →	M	M/I	LIMITS
AD	A	A-D	$\frac{M_c x}{a}$	$\frac{12 M_c c^3 x}{a[ec+(c-x)(f-e)]^3}$	0 - a
DE	D	D-E	$\frac{M_c(b-x)}{b}$	$\frac{12 M_c c^3 (b-x)}{b(hd+ec-hx)^3}$	0 - d
EF	E	E-F	$\frac{M_c(L-c-x)}{b}$	$\frac{12 M_c (L-c-x)}{be^3}$	0 - (L-2c)
FC	F	F-C	$\frac{M_c(c-x)}{b}$	$\frac{12 M_c c^3 (c-x)}{b[ec+(f-e)x]^3}$	0 - c

INTEGRATING BY SECTIONS:

SECTION AD

$$\int_0^a \frac{12 M_c c^3 x \, dx}{a[cf+(e-f)x]^3}$$

$$= \frac{12 M_c c^3}{a h^2}\left[\frac{cf}{2(cf-ah)^2} - \frac{1}{cf-ah} - \frac{1}{2cf} + \frac{1}{cf}\right]$$

$$= \frac{12 M_c c^3}{a h^2}\left[\frac{cf}{2(cf-ah)^2} - \frac{1}{cf-ah} + \frac{1}{2cf}\right] = \frac{6 c^2 M_c a}{f(cf-ah)^2}$$

$$= \frac{6 M_c a}{g^2 f}$$

SECTION DE

$$\int_0^d \frac{12 M_c c^3 (b-x)\, dx}{b(hd+ec-hx)^3}$$

$$= \frac{12 M_c c^3}{b}\left[\frac{bh+j-2hd}{2h^2(j-hd)^2} - \frac{bh+j}{2h^2 j^2}\right]$$

$$= \frac{6 M_c c^3}{bh^2}\left[\frac{bh+j-2hd}{e^2c^2} - \frac{bh+j}{j^2}\right]$$

SECTION EF $\displaystyle\int_0^{L-2c} \frac{12M_c(L-c-x)\,dx}{be^3}$

$$= \frac{12M_c}{be^3}\left[L(L-2c) - c(L-2c) - \frac{(L-2c)^2}{2}\right]$$

$$= \frac{6M_c(L^2-2Lc)}{be^3}$$

SECTION FC $\displaystyle\int_0^{c} \frac{12M_c c^3 (c-x)\,dx}{b[ec+(f-e)x]^3}$

$$= \frac{12M_c c^3}{b}\left[\frac{1}{2h^2c(e+h)} + \frac{h-e}{2h^2e^2c}\right]$$

$$= \frac{6M_c c^2}{h^2 b}\left[\frac{(e-f)^2}{e^2 f}\right] = \frac{6M_c c^2}{be^2 f}$$

SUMMING UP THE SECTIONS:

$$\text{TOTAL AREA } A = \frac{6M_c a}{g^2 f} + \frac{6M_c c^3}{bh^2}\left[\frac{bh+j-2hd}{e^2c^2} - \frac{bh+j}{j^2}\right] + \frac{6M_c(L^2-2Lc)}{be^3} + \frac{6M_c c^2}{be^2 f}$$

FOR DETERMINING THE MOMENT OF THE AREA

ABOUT THE LEFT END OF THE BEAM:

SECTION	X=0	X→	$\frac{M}{I}$	$\frac{Mx}{I}$	LIMITS
AD	A	A-D	$\frac{12M_c c^3 x}{a[ec+(c-x)(f-e)]^3}$	$\frac{12M_c c^3 x^2}{a[ec+(c-x)(f-e)]^3}$	0 – a
DE	D	D-E	$\frac{12M_c c^3(b-x)}{b[hd+ec-hx]^3}$	$\frac{12M_c c^3(b-x)(a+x)}{b[hd+ec-hx]^3}$	0 – d
EF	E	E-F	$\frac{12M_c(L-c-x)}{be^3}$	$\frac{12M_c(L-c-x)(c+x)}{be^3}$	0 – (L-2c)
FC	F	F-C	$\frac{12M_c c^3(c-x)}{b[ec+(f-c)x]^3}$	$\frac{12M_c c^3(c-x)(L-c+x)}{b[ec+(f-e)x]^3}$	0 – c

INTEGRATING BY SECTIONS:

SECTION AD
$$\int_0^a \frac{12M_c c^3 x^2\,dx}{a\left[ec+(c-x)(f-e)\right]^3}$$
$$= \frac{12M_c c^3}{a h^3}\left[\ln\frac{fc}{fc-ha} - \frac{3f^2c^2-4hafc}{2(fc-ha)^2} + \frac{3}{2}\right]$$

SECTION DE
$$\int_0^d \frac{12M_c c^3 (b-x)(a+x)\,dx}{b\left[hd+ec-hx\right]^3}$$

$$= 12M_c c^3 a\left[-\frac{1}{2hj^2} + \frac{1}{2h(ec)^2}\right] + \frac{12M_c c^3}{bh^2}(b-a)\left[\frac{hd-ec}{2(ec)^2} + \frac{1}{2j}\right]$$
$$+\frac{12M_c c^3}{bh^3}\left[\ln\frac{ec}{j} + \frac{3j^2-4jhd}{2(ec)^2} - \frac{3}{2}\right]$$
$$= \frac{6M_c a c}{h}\left[\frac{j^2-(ec)^2}{e^2j^2}\right] + \frac{6M_c c\, d^2(b-a)}{be^2 j}$$
$$+\frac{12M_c c^3}{bh^3}\left[\ln\frac{ec}{j} + \frac{j(3ec-hd)}{2(ec)^2} - \frac{3}{2}\right]$$

SECTION EF
$$\int_0^{L-2c} \frac{12M_c(L-c-x)(c+x)\,dx}{be^3}$$
$$= \frac{12M_c}{be^3}\left[c(L-c)(L-2c) + \frac{(L-2c)^3}{2} - \frac{(L-2c)^3}{3}\right]$$
$$= \frac{12M_c}{be^3}\left[c(L-c)(L-2c) + \frac{(L-2c)^3}{6}\right]$$

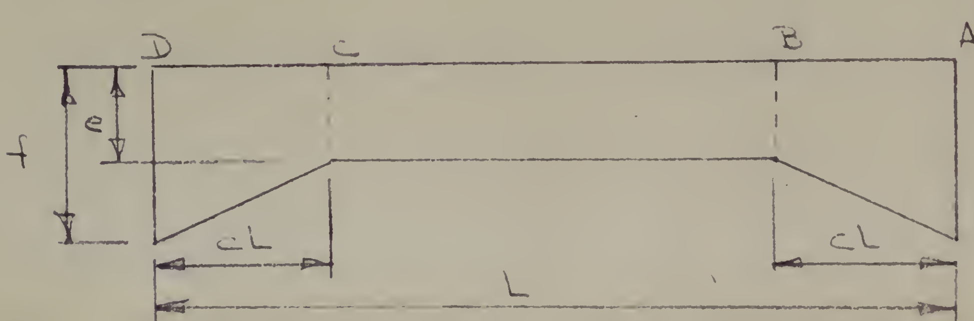
SECTION FC
$$\int_0^c \frac{12M_c c^3 (c-x)(L-c+x)\,dx}{b\left[ec+(f-e)x\right]^3}$$
$$= \frac{6M_c c^2 f(f+e)}{b e^2 f^2} + \frac{6M_c c^2(L-c)}{f^2 e}$$
$$+\frac{12M_c c^3}{bh^3}\left[\ln\frac{e}{f} - \frac{4ef-e^2}{2f^2} + \frac{3}{2}\right]$$

SUMMING UP THE SECTIONS, THE TOTAL MOMENT OF THE AREA OF THE M/I-DIAGRAM ABOUT THE LEFT END OF THE BEAM IS:

$$\bar{M} = \frac{12M_c c^3}{a h^3}\left[\ln \frac{fc}{fc-ha} - \frac{3f^2c^2-4hacf}{2(fc-ha)^2} + \frac{3}{2}\right]$$
$$+ \frac{6M_c ac}{h}\left[\frac{j^2-(ec)^2}{e^2 j^2}\right] + \frac{6M_c c d^2 (b-a)}{b j e^2}$$
$$+ \frac{12M_c c^3}{bh^3}\left[\ln \frac{ec}{j} + \frac{j(3ec-hd)}{2(ec)^2} - \frac{3}{2}\right]$$
$$+ \frac{12M_c}{be^3}\left[c(L-c)(L-2c) + \frac{(L-2c)^3}{c}\right]$$
$$+ \frac{6M_c c^2 g (f+e)}{be^2 f^2} + \frac{6M_c c^2 (L-g)}{f^2 e}$$
$$+ \frac{12M_c c^3}{bh^3}\left[\ln \frac{e}{f} - \frac{4ef-e^2}{2f^2} + \frac{3}{2}\right]$$

THE AREA AND MOMENT OF THE AREA OF THE M/I DIAGRAM OF A BEAM WITH SYMMETRICAL HAUNCHES AT BOTH ENDS AND A MOMENT APPLIED AT ONE END WILL NOW BE DETERMINED. AGAIN, THE BEAM WILL BE CONSIDERED AS BEING SIMPLY SUPPORTED.

CASE IV.



For Determining the Area:

Section	X=0	x →	M	M/I	Limits
AB	A	A-B	$\frac{M_M x}{L}$	$\frac{12 M_M x}{L\left[f-\left(\frac{f-e}{cL}\right)x\right]^3}$	0 - cL
BC	B	B-C	$\frac{M_M (cL+x)}{L}$	$\frac{12 M_M (cL+x)}{Le^3}$	0-(L-2cL)
CD	C	C-D	$\frac{M_M (L-cL+x)}{L}$	$\frac{12 M_M (L-cL+x)}{L\left[e+\left(\frac{f-e}{cL}\right)x\right]^3}$	0-cL

Integrating by Sections:

Section AB $$\int_0^{cL} \frac{12 M_M x \, dx}{L\left[f-\left(\frac{f-e}{cL}\right)x\right]^3}$$

$$= \frac{12 M_M}{L}\left(\frac{cL}{f-e}\right)^2\left[-\frac{1}{e}+\frac{f}{2e^2}+\frac{1}{f}-\frac{1}{2f}\right] = +\frac{6 M_M c^2 L}{fe^2}$$

Section BC $$\int_0^{L-2cL} \frac{12 M_M (cL+x)\,dx}{Le^3}$$

$$= \frac{12 M_M}{Le^3}\left[cL(L-2cL)+\frac{L^2-4cL^2+4c^2L^2}{2}\right]$$

$$= \frac{6 M_M L(1-2c)}{e^3}$$

Section CD $$\int_0^{cL} \frac{12 M_M (L-cL+x)\,dx}{L\left[e+\left(\frac{f-e}{cL}\right)x\right]^3}$$

$$= \frac{6 M_M (L-cL)(cL)(f-e)}{Lf^2e^2} + \frac{12 M_M}{L}\left[\frac{c^2L^2}{(f-e)^2}\right]\left[-\frac{1}{f}+\frac{e}{2f^2}+\frac{1}{2e}\right]$$

$$= \frac{6 M_M (1-c)(cL)(f+e)}{f^2e^2} + \frac{6 M_M c^2 L}{f^2 e}$$

SUMMING UP THE SECTIONS:

TOTAL AREA $A = \frac{6 M_M c^2 L}{f e^2} + \frac{6 M_M L (1-2c)}{e^3}$

$+ \frac{6 M_M (1-c)(cL)(f+e)}{f^2 e^2} + \frac{6 M_M c^2 L}{f^2 e}$

FOR DETERMINING THE TOTAL MOMENT OF THE M/I DIAGRAM ABOUT THE RIGHT END:

SECTION	X=0	X→	M/I	Mx/I	LIMITS
AB	A	A-B	$\frac{12 M_M x}{L\left[f-\left(\frac{f-e}{cL}\right)x\right]^3}$	$\frac{12 M_M x^2}{L\left[f-\left(\frac{f-e}{cL}\right)x\right]^3}$	$0-cL$
BC	B	B-C	$\frac{12 M_M (cL+x)}{Le^3}$	$\frac{12 M_M (cL+x)^2}{Le^3}$	$0-(L-2cL)$
CD	D	D-C	$\frac{12 M_M (L-x)}{L\left[f-\left(\frac{f-e}{cL}\right)x\right]^3}$	$\frac{12 M_M (L-x)^2}{L\left[f-\left(\frac{f-e}{cL}\right)x\right]^3}$	$0-cL$

INTEGRATING BY SECTIONS:

SECTION AB $\int_0^{cL} \frac{12 M_M x^2 dx}{L\left[f-\left(\frac{f-e}{cL}\right)x\right]^3}$

$= \frac{12 M_M}{L}\left[-\frac{c^3 L^3}{(f-e)^3}\right]\left[\ln\frac{e}{f} + \frac{2f}{e} - \frac{f^2}{2e^2} - \frac{3}{2}\right]$

SECTION BC $\int_0^{L-2cL} \frac{12 M_M}{Le^3}\left(c^2L^2 + 2cLx + x^2\right)dx$

$= \frac{12 M_M}{Le^3}\left[c^2L^2(L-2cL) + cL(L-2cL)^2 + \frac{(L-2cL)^3}{3}\right]$

SECTION CD $\int_0^{cL} \frac{12 M_M (L-x)^2 dx}{L\left[f-\left(\frac{f-e}{cL}\right)x\right]^3}$

$= \frac{6 M_M c L^2 (f+e)}{f^2 e^2} - \frac{12 M_M c^2 L^2}{f e^2} - \frac{12 M_M}{L}\left[\frac{c^3 L^3}{(f-e)^3}\right]\left[\ln\frac{e}{f} + \frac{2f}{e} - \frac{f^2}{2e^2} - \frac{3}{2}\right]$

SUMMING UP THE SECTIONS TO OBTAIN THE THE TOTAL MOMENT OF THE M/I DIAGRAM ABOUT THE RIGHT END:

$$M = \frac{12 M_M}{L e^3}\left[c^2L^2(L-2cL) + cL(L-2cL)^2 + \frac{(L-2cL)^3}{3}\right]$$
$$+ \frac{6 M_M c L^2 (f+e)}{f^2 e^2} - \frac{12 M_M c^2 L^2}{f e^2}$$
$$- \frac{24 M_M}{L}\left[\frac{c^3L^3}{(f-e)^3}\right]\left[\ln\frac{e}{f} + \frac{2f}{e} - \frac{f^2}{2e^2} - \frac{3}{2}\right]$$

THE FORMULAE DERIVED ON THE PRECEDING PAGES WERE USED TO MAKE UP THE TABLES PRESENTED IN THE NEXT SECTION. SINCE THE VALUES WHICH OCCUR IN THE TABLES ARE COMPUTED FROM DIRECT INTEGRATIONS, THEY ARE EXACT VALUES AND NOT APPROXIMATE ONES.

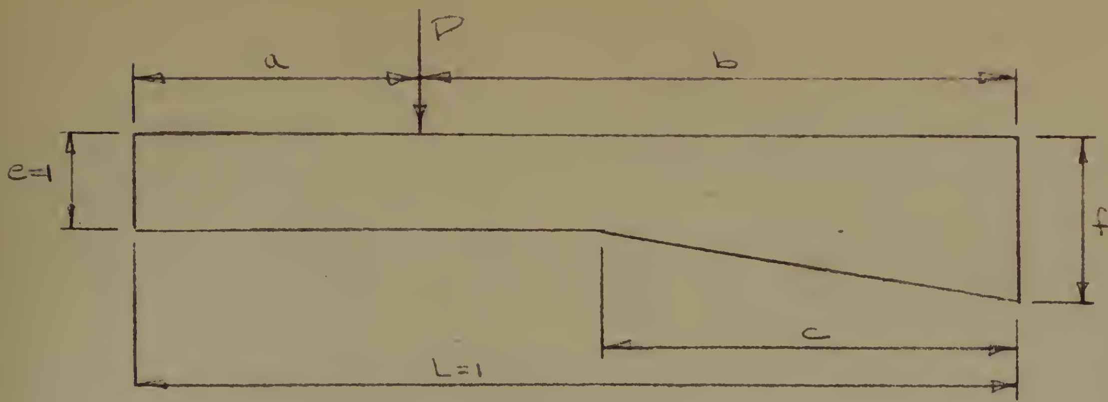
TABLES OF COEFFICIENTS

FOR DETERMINATION OF

AREA & LOCATION OF CENTER OF GRAVITY

OF M OVER EI DIAGRAMS

Haunch at One End — Concentrated Loading



Area M/I diagram $= \dfrac{K_1 L M_c}{e^3}$ where K_1 = Load Coefficient

L = Length

M_c = Max. Simple Moment

e = Minimum Depth

and width of beam is unity.

Distance to center of gravity from left end $= K_a L$

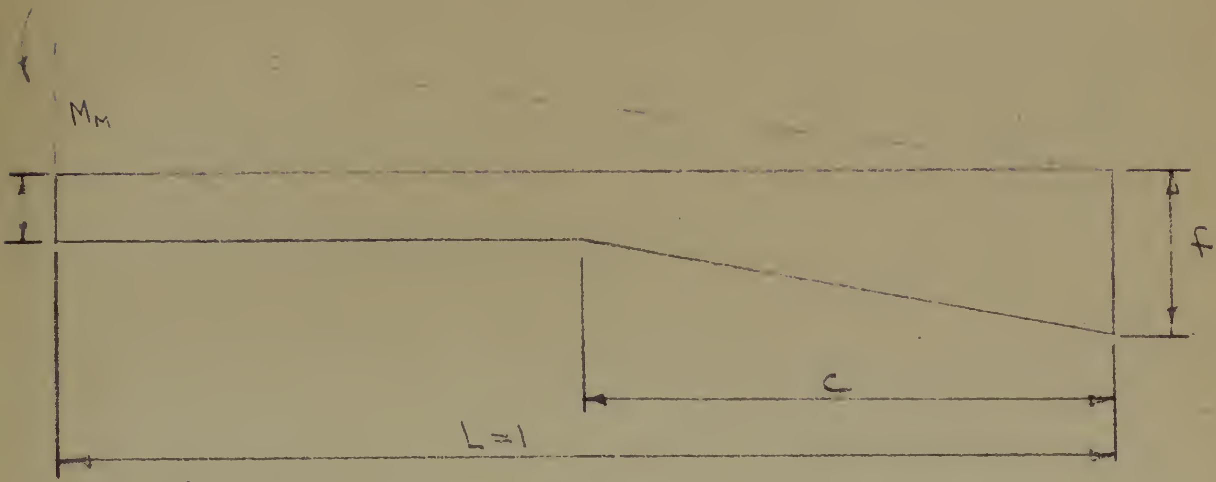
K_1

f/e	c	Load Point a .25	.50	.75
1.5	.25	5.3336	5.7504	5.5008
	.50	5.3336	5.0004	4.4000
	.75	4.5000	3.9643	2.5000
2.0	.25	5.7504	5.6256	5.2512
	.50	5.0000	4.5000	3.6664
	.75	3.1500	3.0001	2.4500
2.5	.25	5.1000	5.5500	5.1000
	.50	4.8000	4.2000	3.2568
	.75	3.3300	2.4444	1.9000
3.0	.25	5.6664	5.4196	4.4492
	.50	4.6664	3.4996	3.0000
	.75	3.0000	2.1000	1.5712

K_a

f/e	c	Load Point a .25	.50	.75
1.5	.25	0.4039	0.4841	0.5576
	.50	0.3772	0.4536	0.5184
	.75	0.3562	0.4387	0.5049
2.0	.25	0.3974	0.4760	0.5437
	.50	0.3561	0.4268	0.4774
	.75	0.3141	0.3939	0.4541
2.5	.25	0.3936	0.4712	0.5352
	.50	0.3500	0.4042	0.4491
	.75	0.2905	0.3603	0.4105
3.0	.25	0.3911	0.4680	0.5295
	.50	0.3340	0.3470	0.4286
	.75	0.2744	0.3346	0.3759

Haunch at One End – Moment at Straight End



$$\text{Area} = \frac{K_2 L M_M}{e^3}$$

K_2

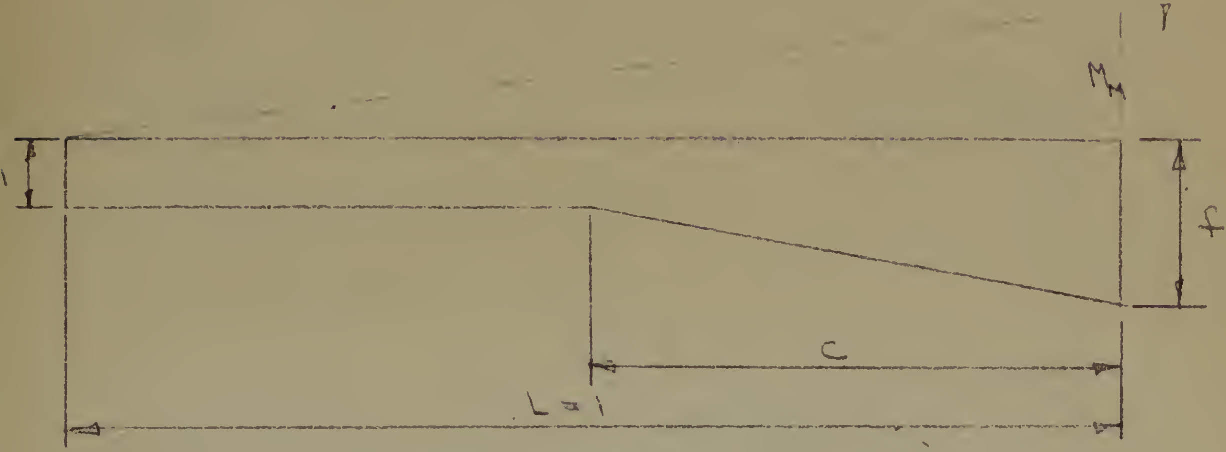
f/e	c .25	.50	.75
1.5	5.3152	5.5002	4.8750
2.0	5.8123	5.2500	4.3125
2.5	5.7750	5.1000	3.9750
3.0	5.7498	4.9998	3.7500

Distance to C.G. from Left End
$= K_b L$

K_b

f/e	c .25	.50	.75
1.5	0.3220	0.2970	0.2720
2.0	0.3164	0.2782	0.2372
2.5	0.3120	0.2665	0.2138
3.0	0.3107	0.2589	0.1963

HAUNCH AT ONE END — MOMENT AT HAUNCHED END



$$\text{AREA} = \frac{K_3 L M_H}{e^3}$$

K_3

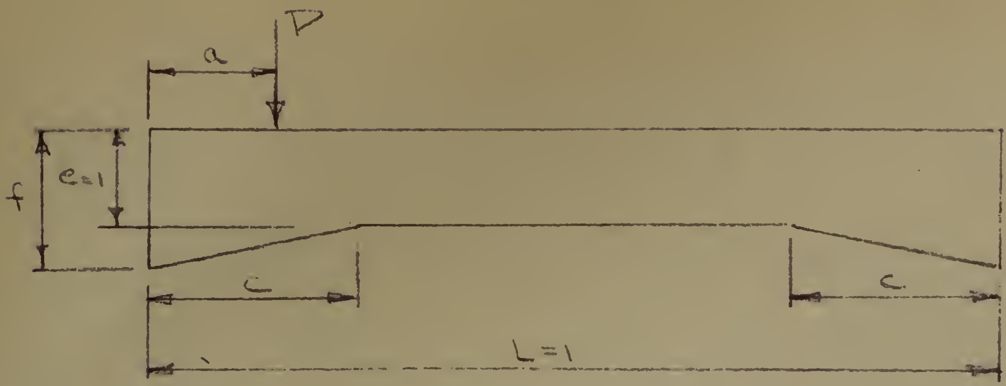
f/e	c = .25	.50	.75
1.5	4.7917	3.8333	3.1250
2.0	4.3125	3.0000	2.0625
2.5	4.0650	2.5900	1.5450
3.0	3.4167	2.3332	1.2500

DISTANCE TO C.G. FROM LEFT END $= K_c L$

K_c

f/e	c = .25	.50	.75
1.5	0.6041	0.5712	0.5648
2.0	0.5736	0.5132	0.5043
2.5	0.5554	0.4731	0.4493
3.0	0.5438	0.4454	0.4060

HAUNCH AT BOTH ENDS - CONCENTRATED LOADING



$$\text{AREA} = \frac{K_4 L M_c}{e^3}$$

K_4

f/e	c	Load Point a: .1667	.3333	.5000
1.5	.1667	5.6000	5.7500	5.7777
	.3333	4.6400	5.0000	5.1112
	.5000	3.6000	3.8571	4.0000
2.0	.1667	5.4000	5.6250	5.6667
	.3333	4.0000	4.5000	4.6667
	.5000	2.5200	2.8150	3.0000
2.5	.1667	5.2800	5.5500	5.5000
	.3333	3.6332	4.2000	4.4000
	.5000	1.9200	2.1449	2.4000
3.0	.1667	5.2000	5.5000	5.5555
	.3333	3.4000	4.0000	4.2222
	.5000	1.5428	1.8000	2.0000

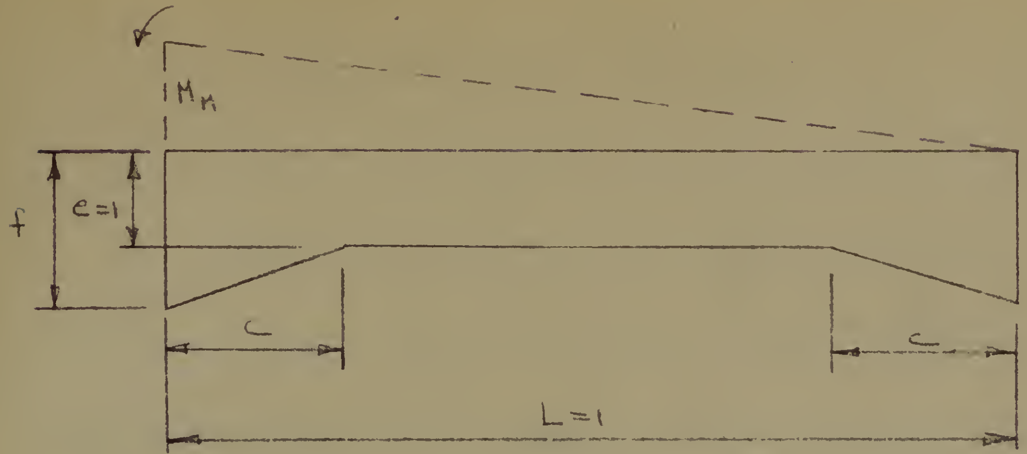
DISTANCE FROM LEFT END TO C.G.

$= K_D L$

K_D

f/e	c	Load Point a: .1667	.3333	.5000
1.5	.1667	0.4006	0.4431	0.5000
	.3333	0.4204	0.4541	0.5000
	.5000	0.4049	0.4553	0.5000
2.0	.1667	0.4066	0.4491	0.5000
	.3333	0.4425	0.4607	0.5000
	.5000	0.4344	0.4637	0.5000
2.5	.1667	0.4104	0.4506	0.5000
	.3333	0.4504	0.4646	0.5000
	.5000	0.4458	0.4647	0.5000
3.0	.1667	0.4130	0.4516	0.5000
	.3333	0.4524	0.4673	0.5000
	.5000	0.4518	0.4738	0.5000

Haunch at Both Ends – Moment at Either End



$$\text{Area} = \frac{K_5 L M_M}{C^3}$$

K_5

f/e	c		
	.1667	.3333	.5000
1.5	5.1111	4.2222	3.3333
2.0	4.7411	3.5000	2.2500
2.5	4.5601	3.1201	1.6800
3.0	4.4444	2.5589	1.2382

Distance to C.G. from Left End

$= K_e L$

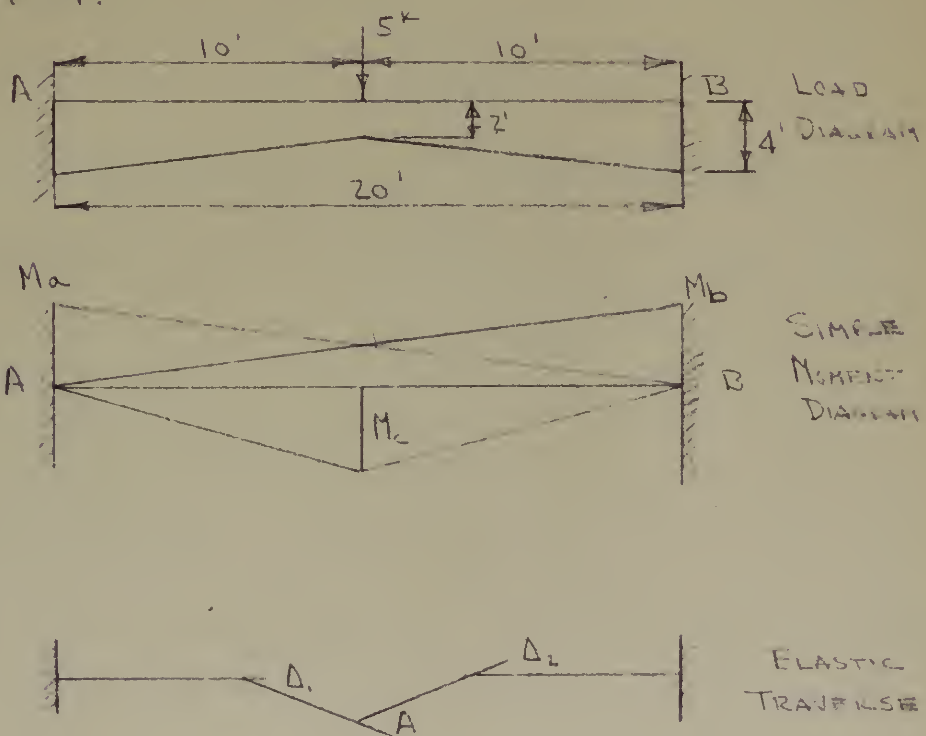
K_e

f/e	c		
	.1667	.3332	.5000
1.5	0.2714	0.3572	0.3804
2.0	0.3811	0.4166	0.4042
2.5	0.3981	0.4334	0.4279
3.0	0.4048	0.4449	0.4411

Illustrative Problems

In order to illustrate the use of the Elastic curve traverse and the tabular coefficients computed in this thesis, several example problems will be worked.

Problem #1.



By the Elastic curve traverse method:

$f/e = 2.0 \quad c = 0.5 \quad \alpha = 0.5$

Then from tables $\quad K_1 = 3000 \quad K_5 = 225$

$K_D = 0500 \quad K_e = 0.492$

$$\Delta_1 = \Delta_2 = \frac{K_5 L M_M}{e^3} = \frac{225 \times 20}{8} M_A = 5.625 M_A$$

$$A = \frac{K_4 L M_c}{e^3} = \frac{K_4 L P a b}{e^3 L} = \frac{3 \times 5 \times 10 \times 10}{8} = 187.5$$

IN ORDER FOR THE TRAVERSE TO CLOSE THE SUM OF THE ANGLES SHOULD EQUAL ZERO.

$$\therefore \quad \Delta_1 - A + \Delta_2 = 0$$

but $\Delta_1 = \Delta_2$

THEN $2\Delta_1 = A$ OR $\Delta_1 = \frac{A}{2} = 93.75$

$$5.625 M_A = 93.75$$

$$\text{OR } M_A = M_B = \frac{93.75}{5.625} = 16.67 \text{ f k.}$$

USING THE <u>HANDBOOK OF FRAME CONSTANTS</u> TO CHECK THE ABOVE SOLUTION, THE TABLE ON PAGE 18 IS USED

$a_A = 0.5 \qquad r_A = 1.0$

$a_B = 0.5 \qquad r_B = 1.0$

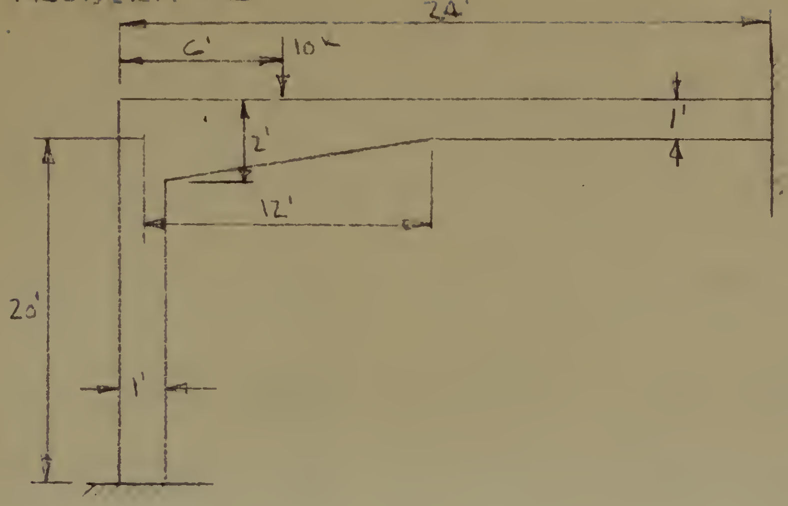
FROM THE TABLE, THE COEFFICIENT FOR DETERMINING THE FIXED END MOMENT IS TAKEN

$$C = .1667$$

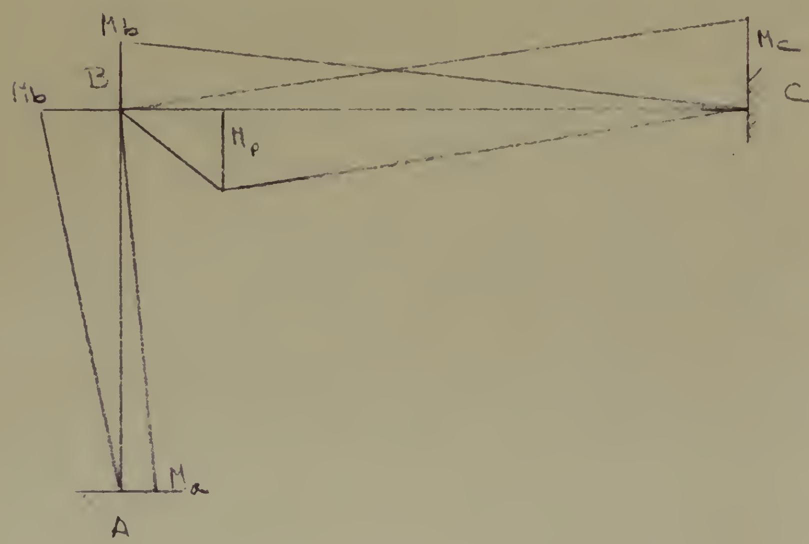
$$F.E.M = M_A = M_B = .1667 \times PL = .1667 \times 5 \times 20$$

$$= 16.67 \text{ f.k (CHECK)}$$

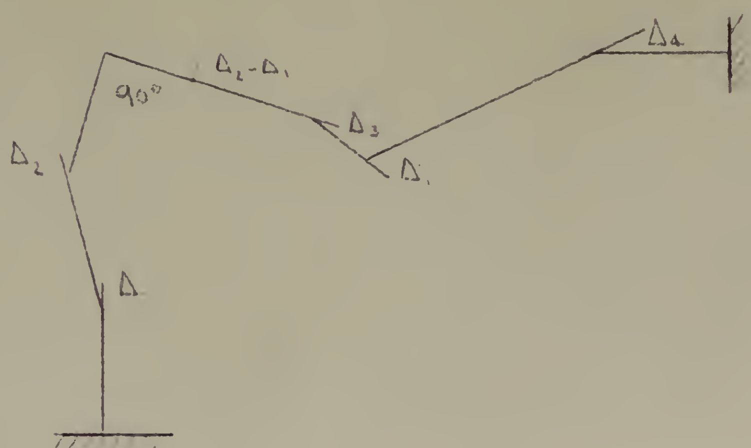
PROBLEM #2



LOAD DIAGRAM



SIMPLE MOMENT DIAGRAM



ELASTIC TRAVERSE

SOLVING BY THE ELASTIC CURVE TRAVERSE METHOD:

CONSIDERING THE MODULUS OF ELASTICITY (E)

AS CONSTANT AND ALL MEMBERS HAVING A WIDTH

OF UNITY, THEN FROM THE TABLES:

$$\Delta_1 = \frac{L_1 M_A}{2I} = 120 M_A \qquad \Delta_3 = \frac{3 L_2 M_B}{e^3} = 72 M_B$$

$$\Delta_2 = \frac{L_1 M_B}{2I} = 120 M_B \qquad \Delta_4 = \frac{5.25 L_2 M_c}{e^3} = 126 M_c$$

$$A_1 = \frac{3.6664 L_2 M_P}{e^3} = 3959.7 \qquad M_A = \frac{M_B}{2}$$

SUMMING UP THE DEFLECTION ANGLES:

$$-\Delta_1 + \Delta_2 + 90^\circ + \Delta_3 - A_1 + \Delta_4 = 90^\circ$$

$$\text{OR } -\Delta_1 + \Delta_2 + \Delta_3 + \Delta_4 = A_1 \qquad (1)$$

SUMMING UP THE DEFLECTIONS FROM B TO C:

$$(\Delta_2 - \Delta_1)L_2 + .5312\,\Delta_3 L_2 - .4779\,A_1 L_2 + .2782\,\Delta_4 L_2 = 0$$

$$\text{OR } (\Delta_2 - \Delta_1) + .5312\,\Delta_3 - .4779\,A_1 + .2782\,\Delta_4 = 0 \qquad (2)$$

SUBSTITUTING THE VALUES FOR Δ:

$$132 M_B + 126 M_c = 3959.7 \qquad (1)$$

$$96.4[illegible] M_B + 35.0[illegible] M_c = 1842.34 \qquad (2)$$

DIVIDING (1) BY 132 AND (2) BY 96.4[illegible]:

$$M_B + .954 M_c = 30.0[illegible] \qquad (1)$$

$$M_B + .362 M_c = 19.52 \qquad (2)$$

SUBTRACTING

$$.592 M_c = 10.48$$

$$\text{OR } M_c = 17.70 \text{ 'K}$$

$$M_B = 19.52 - .362 \times 17.70 = 13.11 \text{ 'K}$$

$$M_A = \frac{13.11}{2} = 6.55 \text{ 'K.}$$

DOING THE SAME PROBLEM BY MOMENT DISTRIBUTION AS A CHECK:

FROM THE HANDBOOK OF FRAME CONSTANTS

$a_A = a_C = 0 \qquad a_B = .50$

$r_A = r_C = 0 \qquad r_B = 1.0$

$C_{CB} = .948 \qquad K_{CB} = 4.99 \qquad K_{CB} = \frac{4.99}{24} = .2079$

$C_{BC} = .385 \qquad K_{BC} = 12.28 \qquad K_{BC} = \frac{12.28}{24} = .5117$

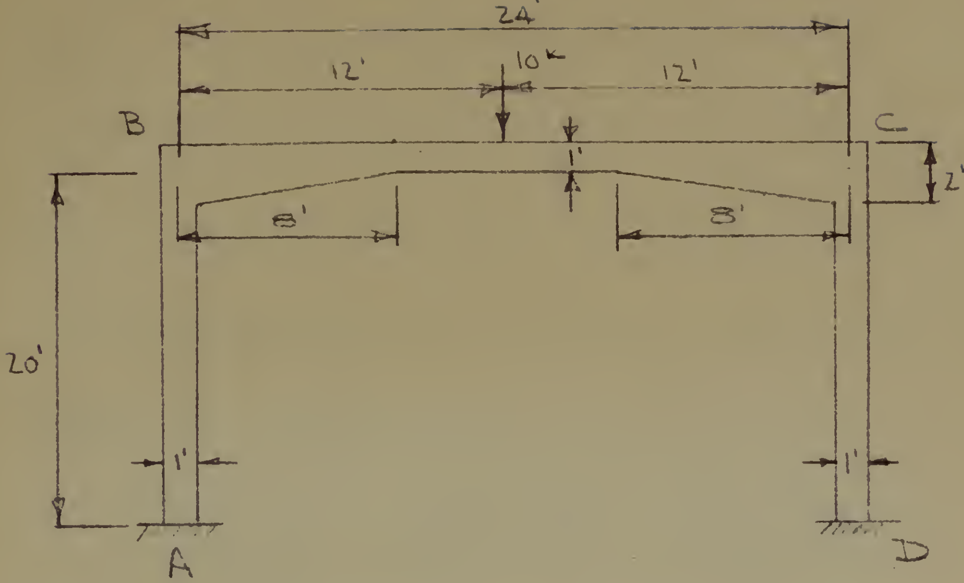
$FEM_{CB} = +.02235 \times 240 = +5.364$

$FEM_{BC} = -.1861 \times 240 = -44.664$

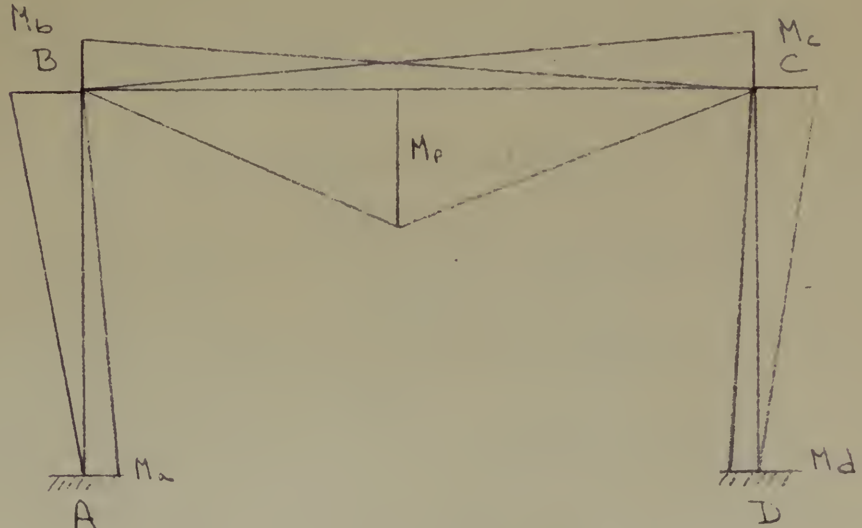
JOINT	A	B		C
MEM.	AB	BA	BC	CB
K	.2000	.2000	.5117	.2079
D.F.	1	.2810	.7190	1
C.F.	.5	.5	.3850	.9480
			−44.664	+5.364
	+6.275	+12.55	+32.113	+12.364
EXTERNAL MOMENTS	+6.275	+12.55	−12.55	+17.728
% ERROR	1.5	5		0.1

AS CAN BE SEEN A FIVE PERCENT ERROR OCCURRED IN THE MOMENT AT B. THIS WAS DUE TO AN INTERPOLATION FROM THE HANDBOOK OF FRAME CONSTANTS. TO GET A MORE ACCURATE CHECK THE VALUES FROM THE HANDBOOK WOULD HAVE TO BE PLOTTED AND THE COEFFICIENT FOR THE FEM TAKEN OFF THE CURVE. THE ABOVE SOLUTION SEEMED ADEQUATE AS A CHECK.

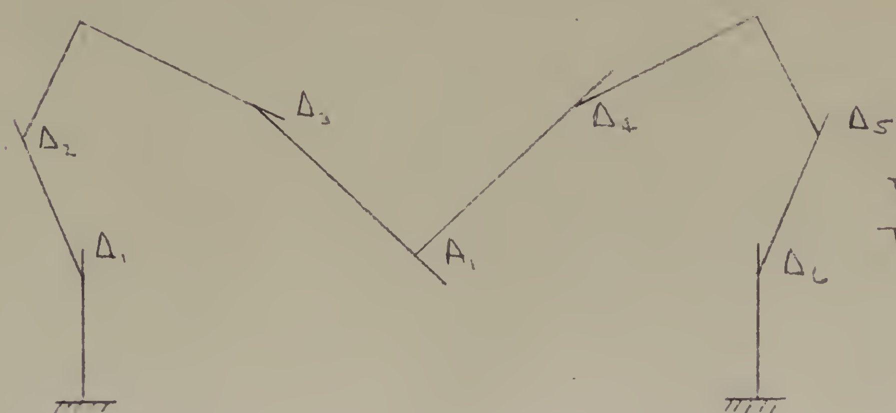
Problem # 3



Load Diagram



Simple Moment Diagram



Elastic Traverse

THE FRAME WILL BE SOLVED FIRST BY THE ELASTIC CURVE TRAVERSE IN TWO WAYS. THE FIRST METHOD WILL INVOLVE A DIRECT SOLUTION. SINCE THE STRUCTURE IS SYMMETRICAL AND IS SYMMETRICALLY LOADED, THE WORK IS GREATLY SIMPLIFIED.

$$\Delta_1 = \Delta_6 \qquad \Delta_2 = \Delta_5 \qquad \Delta_3 = \Delta_4$$
$$M_A = M_D \qquad M_B = M_C \qquad M_A = \frac{M_B}{2}$$

$$\Delta_1 = \Delta_6 = \frac{M_A L_{AB}}{2 I_{AC}} = 120 M_A = 60 M_B$$

$$\Delta_2 = \Delta_5 = \frac{M_B L_{AB}}{2 I_{AB}} = 120 M_B$$

$$\Delta_3 = \Delta_4 = \frac{3.5 L_{BC} M_B}{C^3} = 84 M_B$$

$$A_1 = \frac{4.667 L_{BC} M_0}{C^3} = 6120$$

FOR THE SUM OF THE ANGLES EQUALS 180°:

$$-\Delta_1 + \Delta_2 + \Delta_3 - A_1 + \Delta_4 + \Delta_5 - \Delta_6 + 90 + 90 = 180$$

OR $-2\Delta_1 + 2\Delta_2 + 2\Delta_3 = +A_1$

SUBSTITUTING:

$$-60 M_B + 120 M_B + 84 M_B = \frac{6120}{2}$$
$$144 M_B = 3360$$
$$M_B = 23.33 \text{ fK}$$
$$M_A = \frac{M_B}{2} = 16.67 \text{ fK.}$$

THE SECOND METHOD WILL INVOLVE A RELATIVE SOLUTION. ASSUME Δ_6 TO HAVE A RELATIVE DEFLECTION OF UNITY. THE TRAVERSE WILL BE WORKED BACKWARD TO FIND THE RELATIVE A_1. THE ACTUAL Δ_6 CAN BE FOUND BY THE RELATIONSHIP $\frac{A_1(\text{ACTUAL})}{A_1(\text{REL})} \times \Delta_6(\text{REL}) = \Delta_6(\text{ACTUAL})$

SO LET $\Delta_6 = 1$

DEFLECTION OF $C = 0 \quad \therefore \; 2\Delta_6 - \Delta_5 = 0$

OR $\Delta_5 = 2$

SLOPE AT $C = \Delta_5 - \Delta_6 = 1$

DEFLECTION FROM C TO B IS ZERO:

$$\therefore \; 1 + (1 - .4166)\Delta_4 - .5A_1 + .4166\Delta_3 = 0$$

BY SYMMETRY $\Delta_4 = \Delta_3$

$$\therefore \; 1 + \Delta_{4(\text{REL})} - .5A_1 = 0$$

$$A_{1(\text{REL})} = 2(1 + \Delta_{4(\text{REL})})$$

$$\Delta_{4(\text{REL})} = \frac{3.5 \times 24 \times M_{c(\text{REL})}}{e^2} = 84 M_{c(\text{REL})}$$

$$M_c(\text{REL}) = 2M_{D(\text{REL})} = \frac{2 \times 2}{12 \times 20} = \frac{1}{60}$$

$$\therefore \; A_1(\text{REL}) = 2\left(1 + \frac{84}{60}\right) = 2(2.4) = 4.8$$

$$A_1(\text{ACTUAL}) = 6720$$

$$\therefore \; \Delta_6(\text{ACTUAL}) = \frac{6720}{4.8} = 1400$$

AND $M_D = M_A = \frac{1400 \times 1 \times 2}{12 \times 20} = 11.67$ f k

$$M_c = M_b = 2M_D = 23.33 \text{ f.k}$$

THE PROBLEM WILL NOW BE CHECKED BY MOMENT DISTRIBUTION. AFTER PICKING THE PROPER VALUE OF THE COEFFICIENTS AND THE F.E.M.'S FROM THE TABLES, THE FOLLOWING DISTRIBUTION RESULTS.

JOINT	A	B		C		D
MEM.	AB	BA	BC	CB	CD	DC
K	.2000	.2000	.4496	.4496	2000	2000
D.F.	1	.2859	.7141	.7141	.2859	1
C.F.	.5	.5	.710	.710	.5	.5
			-39.84	+39.84		
	+5.69	+11.39	+28.45	+20.20		
			-30.42	-42.95	-17.16	-8.58
	+4.35	+8.70	+21.72	+15.42		
			-7.92	-11.01	-4.41	-2.205
	+1.12	+2.24	+5.58	+3.96		
			-2.01	-2.83	-1.13	-.565
	+.285	+.57	+1.44	+1.02		
			-.52	-.73	-.29	-.145
	+.075	+.15	+.37	+.26		
			-.13	-.19	-.07	-.035
	+.02	+.04	+.09	+.06		
			-.03	-.04	-.02	-.01
	+.005	+.01	+.02			
EXTERNAL MOMENTS	+11.54	+23.10	-23.10	+23.11	-23.08	-11.54

NOTE THAT IN ALL PROBLEMS THE SIMPLE MOMENT DIAGRAM WAS DRAWN ON THE TENSION SIDE OF THE MEMBER. THIS WAS DONE FOR THE FOLLOWING REASON. IF THE SIMPLE MOMENT DIAGRAM IS CONSIDERED TO ACT AS A CONCENTRATED LOAD THROUGH ITS CENTER OF

GRAVITY, THEN THE DEFLECTION ANGLE AT THAT CENTER OF GRAVITY WILL BE IN THE DIRECTION OF THE FICTITIOUS LOAD PUSHING AGAINST THE BEAM. THIS ARBITRARY CONVENTION CAN BE CHECKED EASILY IN THE FIRST SAMPLE PROBLEM. THIS IS THE ONLY REASON WHY THE SIMPLE MOMENT DIAGRAMS HAVE BEEN DRAWN.

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